PERFORMANCE ESTIMATION OF FIRST-ORDER METHODS INVOLVING LINEAR MAPPINGS

Nizar Bousselmi

François Glineur and Julien Hendrickx

Institute of Information and Communication Technologies, Electronics and Applied Mathematics (ICTEAM)
Université catholique de Louvain (UCLouvain)

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Worst-case **performance** of an optimization **method** $\mathcal{M}$ on

$$\min_x f(x)$$

where $f \in \mathcal{F}$ has some properties (smoothness, convexity, ...)?

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Example 1:

Worst-case performance of $\overbrace{\text{gradient method}}^{\mathcal{M}}$ on $\overbrace{L\text{-smooth convex functions}}^{\mathcal{F}}$ after N iterations?

$$\overbrace{f(x_N) - f^*}^{\text{performance}} \leq \frac{L}{2}\frac{1}{2N+1}.$$

Worst-case **performance** of an optimization **method** $\mathcal{M}$ on

$$\min_x f(x) + g(Mx)$$

where $f \in \mathcal{F}$ has some properties (smoothness, convexity, ...)?

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$$\overbrace{f(x_N) - f^*}^{\text{performance}} \leq \frac{L}{2} \frac{1}{2N+1}.$$

Example 2:

Worst-case performance of methods involving linear mappings?

e.g. Chambolle-Pock method, Condat-Vũ method, PDFP, PD3O, PAPC, ADMM, etc.

Performance Estimation Problem (PEP)

Theoretical and practical framework to analyze **performance** of optimization methods on problem classes.

- *Performance of first-order methods for smooth convex minimization: a novel approach* [Drori and Teboulle, 2014]
- *Convex interpolation and performance estimation of first-order methods for convex optimization* [Taylor, 2017]

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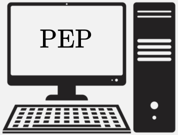
Method
+
Class of functions



Exact worst-case performance

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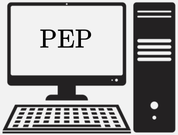
Gradient method

+

Class of L-smooth convex functions



Exact worst-case performance

$$f(x_N) - f^* \leq \frac{L}{2}\frac{1}{2N+1}$$

PEP IS WELL DEVELOPED: AROUND 40 RESEARCHERS

Analysis of methods and frameworks

- Decentralized optimization [Colla, Hendrickx, 2022]
- Halper iteration *[Lieder, 2021]*
- Maximal monotone inclusion *[Gu, Yang, 2020]*
- Proximal GM *[Taylor, Hendrickx, Glineur, 2018]*
- Splitting methods *[Ryu, Taylor, Bergeling, Giselsson 2020]*
- Non linear conjugate gradients [Gupta, Freund, Sun, Taylor, 2023]
- Monotone variational inequality *[Gu and Yang, 2022]*
- GM on smooth hypoconvex *[Rotaru, Glineur, Patrinos, 2022]*
- Stochastic optimization *[Taylor, Bach, 2021]*
- Difference of convex method *[Abbaszadehpeivasti, de Klerk, Zamani, 2021]*
- ADMM [Abbaszadehpeivasti, de Klerk, Zamani, 2022]
- GM on smooth strongly convex *[Taylor, Hendrickx, Glineur, 2017]*
- GM on smooth convex *[Drori and Teboulle, 2014]*
- GM exact line search *[de Klerk, Glineur, Taylor, 2017]*
- Extragradient method *[Gorbunov, Loizou, Gidel, 2020]*
- Coordinate descent method *[Shi, Liu, 2016]* *[Kamri, Hendrickx, Glineur, 2022]* *[Abbaszadehpeivasti, de Klerk, Zamani, 2022]*
- Continuous-time models *[Moucer, Taylor, Bach, 2022]*
- Gradient descent-ascent method [Zamani, de Klerk, Abbaszadehpeivasti, 2022]
- PESTO *[Taylor, Hendrickx, Glineur, 2017]*
- Inexact GM *[de Klerk, Glineur, Taylor, 2020]*
- PEPit *[Goujaud, Moucer, Glineur, Hendrickx, Taylor, Dieuleveut, 2022]*
- Methods on convex functions with quadratic upper bound *[Goujaud, Taylor, Dieuleveut, 2022]*
- Bregman methods *[Dragomir, Taylor, d'Aspremont, 2021]*
- GM on functions with lower restricted secant inequality and upper error bound *[Guille-Escuret, Ibrahim, Goujaud, Mitliagkas, 2022]*

Optimization of methods

- Optimal GM for smooth strongly convex *[Taylor, Drori, 2022]*
- Optimal methods for non-smooth and smooth convex *[Drori, Taylor, 2020]*
- Fast proximal point method *[Kim, 2021]*
- Kelley's cutting-plane method *[Drori, Teboulle, 2016]*
- Optimized GM for smooth convex *[Kim, Fessler, 2016]*
- FISTA *[Kim, Fessler, 2018]*
- Optimal first-order methods for convex and nonconvex *[Gupta, Van Parys, Ryu, 2022]*

GM = Gradient method   ADMM = Alternating Directions Method of Multipliers

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- **Functions involving linear mappings** *[Bousselmi, Hendrickx, Glineur, 2023]*
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Performance Estimation Problem

Input = optimization method + problem class
Output = worst instance in problem class

Example:

- N steps of gradient method $x_{i+1} = x_i - \frac{1}{L}\nabla f(x_i)$
- L-smooth convex functions f

PEP

$$
\begin{aligned}
\max_{\text{points } x_i, x^*, \text{ function } f} \quad & f(x_N) - f(x^*) \\
\text{s.t.} \quad & f \ L\text{-smooth convex}, \\
& x_{i+1} = x_i - \frac{1}{L}\nabla f(x_i), \\
& ||x^* - x_0||^2 \leq 1, \\
& ||\nabla f(x^*)||^2 = 0.
\end{aligned}
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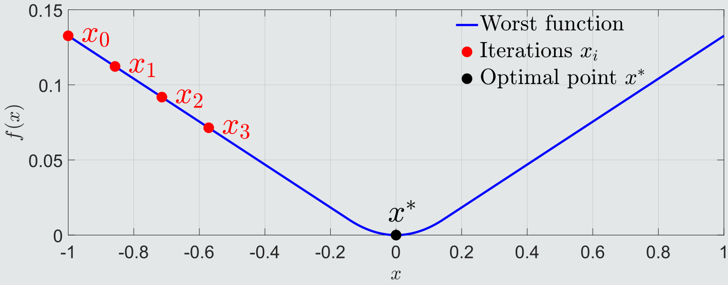
PEP solved outputs

- Worst performance: $\boldsymbol{f(x_N) - f^*} \leq \frac{L}{2}\frac{1}{2N+1}$ (for all N and L);
- Worst function: (for $N = 3$ and $L = 1$)



PEP as finite-dimensional problem

f **infinite-dimensional** but only access to x_i, $f(x_i)$, $\nabla f(x_i)$ … **black-box** property!

PEP

$$\max_{\text{points } x_i, x^*, \text{function } f} \quad f(x_N) - f(x^*)$$

$$\text{s.t.} \quad f \ L\text{-smooth convex},$$

$$x_{i+1} = x_i - \frac{1}{L}\nabla f(x_i),$$

$$||x^* - x_0||^2 \leq 1,$$

$$||\nabla f(x^*)||^2 = 0.$$

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PEP

$$\max_{\text{points } x_i, x^*, f_i, f^*, g_i, g^*} \quad f_N - f^*$$

$$\text{s.t.} \quad \exists f \; L\text{-smooth convex}: \quad f(x_i) = f_i, \quad \nabla f(x_i) = g_i,$$

$$f(x^*) = f^*, \quad \nabla f(x^*) = g^*,$$

$$x_{i+1} = x_i - \frac{1}{L} g_i,$$

$$||x^* - x_0||^2 \leq 1,$$

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Interpolation condition to reformulate.

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$$\max_{\text{points } x_i, x^*, f_i, f^*, g_i, g^*} \quad f_N - f^*$$

$$\text{s.t.} \quad f_i \geq f_j + g_j^T(x_i - x_j) + \frac{1}{2L}||g_i - g_j||^2,$$

$$x_{i+1} = x_i - \frac{1}{L} g_i,$$

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Interpolation condition to reformulate.

Can be reformulated as convex **semidefinite problem**, efficiently solvable!

Interpolation conditions for L-smooth convex functions

Given x_i, g_i, f_i $\quad \forall i = 0, \ldots, N$,

$\exists$ L-smooth convex f such that $\begin{cases} f(x_i) & = f_i \;\; \forall i = 0, \ldots, N, \\ \nabla f(x_i) & = g_i \;\; \forall i = 0, \ldots, N, \end{cases}$ if, and only if,

$$f_i \geq f_j + g_j^T (x_i - x_j) + \frac{1}{2L} ||g_i - g_j||^2 \quad \forall i, j = 0, \ldots, N.$$

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$$\underbrace{f_i \geq f_j + g_j^T(x_i - x_j) + \frac{1}{2L}||g_i - g_j||^2}_{\text{Linear in } f_i,\ x_i^T g_j,\ x_i^T x_j,\ g_i^T g_j} \quad \forall i, j = 0, \dots, N.$$

Interpolation conditions must be convex in $f_i, \quad x_i^T g_j, \quad x_i^T x_j, \quad g_i^T g_j$.

Remark: Interpolation conditions and PEP formulation available for numerous function classes: **non-smooth, L-smooth, convex, μ-strongly convex, etc.**

Performance Estimation Problem for linear mappings

Goal: Extend and exploit PEP to analyze methods applied to $\min_x F(x)$ where F involves linear mappings.

$F(x)$	Possible method	Iterations
$g(Mx)$	Gradient descent	$x_{i+1} = x_i - \frac{h}{L} M^T \nabla g(Mx_i)$
$\frac{1}{2} x^T Q x \ (= \frac{1}{2} \lVert Q^{\frac{1}{2}} x \rVert^2)$	Gradient descent	$x_{i+1} = x_i - \frac{h}{L} Q x_i$
$f(x) + g(Mx)$	Chambolle-Pock [Chambolle and Pock, 2011]	$\begin{cases} x_{i+1} & = \mathrm{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} & = \mathrm{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$

What is missing in PEP to analyze such problems?

Goal: Extend and exploit PEP to analyze methods applied to $\min_x F(x)$ where F involves linear mappings.

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What is missing in PEP to analyze such problems?

Interpolation conditions for linear mappings!

Decomposing the iteration

- Gradient method on $\min_x g(Mx)$: $x_{i+1} = x_i - \frac{h}{L} M^T \nabla g(Mx_i)$

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- Gradient method on $\min_x g(Mx)$: $x_{i+1} = x_i - \frac{h}{L} M^T \nabla g(Mx_i)$ or equivalently

$$\begin{cases} y_i & = Mx_i \\ u_i & = \nabla g(y_i) \\ v_i & = M^T u_i \\ x_{i+1} & = x_i - \frac{h}{L} v_i \end{cases}$$

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$$\begin{cases} y_i & = Mx_i & \text{New interpolation conditions} \\ u_i & = \nabla g(y_i) & \text{Standard interpolation conditions} \\ v_i & = M^T u_i & \text{New interpolation conditions} \\ x_{i+1} & = x_i - \frac{h}{L} v_i & \text{Standard} \end{cases}$$

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- Chambolle-Pock method on $\min_x f(x) + g(Mx)$:

$$\begin{cases} x_{i+1} & = \mathrm{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} & = \mathrm{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$$

Also requires New interpolation conditions.

Definition (L-matrix-interpolability)

$\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ are **L-matrix-interpolable** if, and only if,

$$\exists M \text{ with } \sigma_{\max}(M) \leq L \ : \ \begin{cases} y_i = M x_i, & \forall i = 1, \dots, N_1, \\ v_j = M^T u_j, & \forall j = 1, \dots, N_2. \end{cases}$$

Interpolation conditions as semidefinite constraints

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Theorem (L-matrix-interpolation conditions)

*$\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ are **L-matrix-interpolable** if, and only if,*

$$\begin{cases} X^T V = Y^T U, \\ Y^T Y \preceq L^2 X^T X, \\ V^T V \preceq L^2 U^T U, \end{cases}$$

where $X = (x_1 \ \cdots \ x_{N_1})$, $Y = (y_1 \ \cdots \ y_{N_1})$, $U = (u_1 \ \cdots \ u_{N_2})$ and $V = (v_1 \ \cdots \ v_{N_2})$.

Remark: [Colla and Hendrickx, 2022] and [Abbaszadehpeivasti et al., 2022] used these conditions knowing their necessity.

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Proof.
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$$\Leftrightarrow \exists M \text{ with } M^T M \preceq L^2 I \text{ and } M M^T \preceq L^2 I \ : \ Y = MX \text{ and } V = M^T U$$

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where $X = (x_1 \ \cdots \ x_{N_1})$, $Y = (y_1 \ \cdots \ y_{N_1})$, $U = (u_1 \ \cdots \ u_{N_2})$ and $V = (v_1 \ \cdots \ v_{N_2})$.

Proof.
Necessity: Let $\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ L-matrix-interpolable.

$$\Leftrightarrow \exists M \text{ with } \sigma_{\max}(M) \leq L \ : \quad y_i = Mx_i \text{ and } v_j = M^T u_j \ \ \forall i, j$$

$$\Leftrightarrow \exists M \text{ with } M^T M \preceq L^2 I \text{ and } MM^T \preceq L^2 I \ : \ Y = MX \text{ and } V = M^T U$$

Theorem (L-matrix-interpolation conditions)

*$\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ are **L-matrix-interpolable** if, and only if,*

$$\begin{cases} X^TV = Y^TU, \\ Y^TY \preceq L^2X^TX, \\ V^TV \preceq L^2U^TU, \end{cases}$$

where $X = (x_1 \ \cdots \ x_{N_1})$, $Y = (y_1 \ \cdots \ y_{N_1})$, $U = (u_1 \ \cdots \ u_{N_2})$ and $V = (v_1 \ \cdots \ v_{N_2})$.

Proof.
Necessity: Let $\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ L-matrix-interpolable.

$$\Leftrightarrow \exists M \text{ with } \sigma_{\max}(M) \leq L \ : \quad y_i = Mx_i \text{ and } v_j = M^Tu_j \ \ \forall i, j$$

$$\Leftrightarrow \exists M \text{ with } M^TM \preceq L^2I \text{ and } MM^T \preceq L^2I \ : \ Y = MX \text{ and } V = M^TU$$

$$\Rightarrow \underbrace{X^TM^TU}_{X^TV} = \underbrace{X^TM^TU}_{Y^TU}, \qquad \underbrace{X^TM^TMX}_{Y^TY} \preceq L^2X^TX, \qquad \underbrace{U^TMM^TU}_{V^TV} \preceq L^2U^TU$$

Theorem (L-matrix-interpolation conditions)

*$\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ are **L-matrix-interpolable** if, and only if,*

$$\begin{cases} X^T V = Y^T U, \\ Y^T Y \preceq L^2 X^T X, \\ V^T V \preceq L^2 U^T U, \end{cases}$$

where $X = (x_1 \; \cdots \; x_{N_1})$, $Y = (y_1 \; \cdots \; y_{N_1})$, $U = (u_1 \; \cdots \; u_{N_2})$ and $V = (v_1 \; \cdots \; v_{N_2})$.

Proof.
Necessity: Let $\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ L-matrix-interpolable.

$$\Leftrightarrow \exists M \text{ with } \sigma_{\max}(M) \leq L \;:\quad y_i = Mx_i \text{ and } v_j = M^T u_j \;\; \forall i, j$$

$$\Leftrightarrow \exists M \text{ with } M^T M \preceq L^2 I \text{ and } MM^T \preceq L^2 I \;:\; Y = MX \text{ and } V = M^T U$$

$$\Rightarrow \underbrace{X^T M^T U}_{X^T V} = \underbrace{X^T M^T U}_{Y^T U}, \qquad \underbrace{X^T M^T M X}_{Y^T Y} \preceq L^2 X^T X, \qquad \underbrace{U^T M M^T U}_{V^T V} \preceq L^2 U^T U$$

Theorem (L-matrix-interpolation conditions)

*$\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ are **L-matrix-interpolable** if, and only if,*

$$\begin{cases} X^T V = Y^T U, \\ Y^T Y \preceq L^2 X^T X, \\ V^T V \preceq L^2 U^T U, \end{cases}$$

where $X = (x_1 \ \cdots \ x_{N_1})$, $Y = (y_1 \ \cdots \ y_{N_1})$, $U = (u_1 \ \cdots \ u_{N_2})$ and $V = (v_1 \ \cdots \ v_{N_2})$.

Proof.
Necessity: Let $\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ L-matrix-interpolable.

$$\Leftrightarrow \exists M \text{ with } \sigma_{\max}(M) \leq L \ : \quad y_i = Mx_i \text{ and } v_j = M^T u_j \ \ \forall i, j$$

$$\Leftrightarrow \exists M \text{ with } M^T M \preceq L^2 I \text{ and } MM^T \preceq L^2 I \ : \ Y = MX \text{ and } V = M^T U$$

$$\Rightarrow \underbrace{X^T M^T U}_{X^T V} = \underbrace{X^T M^T U}_{Y^T U}, \qquad \underbrace{X^T M^T M X}_{Y^T Y} \preceq L^2 X^T X, \qquad \underbrace{U^T M M^T U}_{V^T V} \preceq L^2 U^T U$$

Theorem (L-matrix-interpolation conditions)

*$\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ are **L-matrix-interpolable** if, and only if,*

$$\begin{cases} X^T V = Y^T U, \\ Y^T Y \preceq L^2 X^T X, \\ V^T V \preceq L^2 U^T U, \end{cases}$$

where $X = (x_1 \ \cdots \ x_{N_1})$, $Y = (y_1 \ \cdots \ y_{N_1})$, $U = (u_1 \ \cdots \ u_{N_2})$ and $V = (v_1 \ \cdots \ v_{N_2})$.

Proof.
Necessity: Let $\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ L-matrix-interpolable.

$$\begin{aligned} &\Leftrightarrow \exists M \text{ with } \sigma_{\max}(M) \le L \ : \quad y_i = Mx_i \text{ and } v_j = M^T u_j \ \ \forall i, j \\ &\Leftrightarrow \exists M \text{ with } M^T M \preceq L^2 I \text{ and } MM^T \preceq L^2 I \ : \ Y = MX \text{ and } V = M^T U \\ &\Rightarrow \underbrace{X^T M^T U}_{X^T V} = \underbrace{X^T M^T U}_{Y^T U}, \quad \underbrace{X^T M^T M X}_{Y^T Y} \preceq L^2 X^T X, \quad \underbrace{U^T M M^T U}_{V^T V} \preceq L^2 U^T U \end{aligned}$$

Sufficiency: see [Bousselmi, Hendrickx, Glineur, 2023]. □

Definition ($(0, L)$-symmetric-matrix-interpolability)

$\{(x_i, y_i)\}_{i=1,\dots,N}$ is **$(0, L)$-symmetric-matrix-interpolable** if, and only if,

$$\exists Q \text{ symmetric with } 0 \preceq Q \preceq LI \ : \ y_i = Qx_i \quad \forall i = 1, \dots, N.$$

Theorem ($(0, L)$-symmetric-matrix-interpolation conditions)

*$\{(x_i, y_i)\}_{i=1,\dots,N}$ is **$(0, L)$-symmetric-matrix-interpolable** if, and only if,*

$$\begin{cases} X^T Y = Y^T X, \\ Y^T (LX - Y) \succeq 0, \end{cases}$$

where $X = (x_1 \ \cdots \ x_{N_1})$ and $Y = (y_1 \ \cdots \ y_{N_1})$.

Remark: Similar result for skew-symmetric matrices.

Definition ((μ, L)-symmetric-matrix-interpolability)

$\{(x_i, y_i)\}_{i=1,\ldots,N}$ is **(μ, L)-symmetric-matrix-interpolable** if, and only if,

$$\exists Q \text{ symmetric with } \mu I \preceq Q \preceq LI \;:\; y_i = Qx_i \quad \forall i = 1, \ldots, N.$$

Theorem ((μ, L)-symmetric-matrix-interpolation conditions)

*$\{(x_i, y_i)\}_{i=1,\ldots,N}$ is **(μ, L)-symmetric-matrix-interpolable** if, and only if,*

$$\begin{cases} X^T Y = Y^T X, \\ (Y - \mu X)^T (LX - Y) \succeq 0, \end{cases}$$

where $X = (x_1 \;\cdots\; x_{N_1})$ and $Y = (y_1 \;\cdots\; y_{N_1})$.

Remark: Similar result for skew-symmetric matrices.

Homogeneous quadratic functions: $\mathcal{Q}_{\mu,L} = \{f(x) = \frac{1}{2}x^TQx, Q = Q^T, \mu I \preceq Q \preceq LI\}$

Definition ($\mathcal{Q}_{\mu,L}$-interpolability)

$\{(x_i, g_i, f_i)\}_{i=1,\ldots,N}$ is **$\mathcal{Q}_{\mu,L}$-interpolable** if, and only if,

$$\exists Q \text{ symmetric with } \mu I \preceq Q \preceq LI \ : \ \begin{cases} f_i = \frac{1}{2}x_i^T Q x_i, \\ g_i = Q x_i, \end{cases} \quad \forall i = 1, \ldots, N.$$

Homogeneous quadratic functions: $\mathcal{Q}_{\mu,L} = \{f(x) = \frac{1}{2}x^TQx, Q = Q^T, \mu I \preceq Q \preceq LI\}$

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Theorem ($\mathcal{Q}_{\mu,L}$-interpolation conditions)

*$\{(x_i, g_i, f_i)\}_{i=1,\ldots,N}$ is **$\mathcal{Q}_{\mu,L}$-interpolable** if, and only if,*

$$\begin{cases} X^TG = G^TX, \\ (G - \mu X)^T(LX - G) \succeq 0, \\ f_i = \frac{1}{2}x_i^Tg_i \quad \forall i = 1, \ldots, N, \end{cases}$$

where $X = (x_1 \ \cdots \ x_N)$ and $G = (g_1 \ \cdots \ g_N)$.

Homogeneous quadratic functions: $\mathcal{Q}_{\mu,L} = \{f(x) = \frac{1}{2}x^T Q x, Q = Q^T, \mu I \preceq Q \preceq LI\}$

Definition ($\mathcal{Q}_{\mu,L}$-interpolability)

$\{(x_i, g_i, f_i)\}_{i=1,\dots,N}$ is **$\mathcal{Q}_{\mu,L}$-interpolable** if, and only if,

$$\exists Q \text{ symmetric with } \mu I \preceq Q \preceq LI \ : \ \begin{cases} f_i = \frac{1}{2}x_i^T Q x_i, \\ g_i = Q x_i, \end{cases} \quad \forall i = 1, \dots, N.$$

Theorem ($\mathcal{Q}_{\mu,L}$-interpolation conditions)

*$\{(x_i, g_i, f_i)\}_{i=1,\dots,N}$ is **$\mathcal{Q}_{\mu,L}$-interpolable** if, and only if,*

$$\begin{cases} X^T G = G^T X, \\ (G - \mu X)^T (LX - G) \succeq 0, \\ f_i = \frac{1}{2}x_i^T g_i \quad \forall i = 1, \dots, N, \end{cases}$$

where $X = (x_1 \ \cdots \ x_N)$ and $G = (g_1 \ \cdots \ g_N)$.

Allows to use PEP on the class of quadratic functions!

Exploitation of matrix-interpolability

1 Gradient method on $g(Mx)$

2 Gradient method on $\frac{1}{2}x^TQx$

3 Chambolle-Pock method on $f(x) + g(Mx)$

Simplest problem $\min_x F(x)$ involving linear mapping

Function class: $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx), g \text{ is } L_g\text{-smooth } \mu_g\text{-strongly convex}, 0 \leq ||M|| \leq L_M\}$

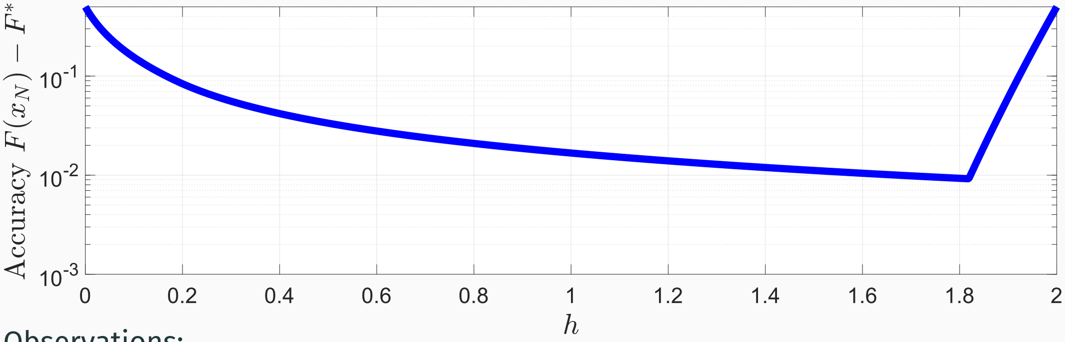
Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L_g L_M^2} \nabla F(x_i)$

Function class: $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx),\, g \text{ is } L_g\text{-smooth } \mu_g\text{-strongly convex},\, 0 \leq ||M|| \leq L_M\}$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L_g L_M^2} \nabla F(x_i)$

Parameters: $L_g = L_M = 1$, $\mu_g = 0.1$, $\kappa_g = \frac{\mu_g}{L_g}$, $||x_0 - x^*||^2 \leq 1$, $h_0(\kappa_g, N)$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$



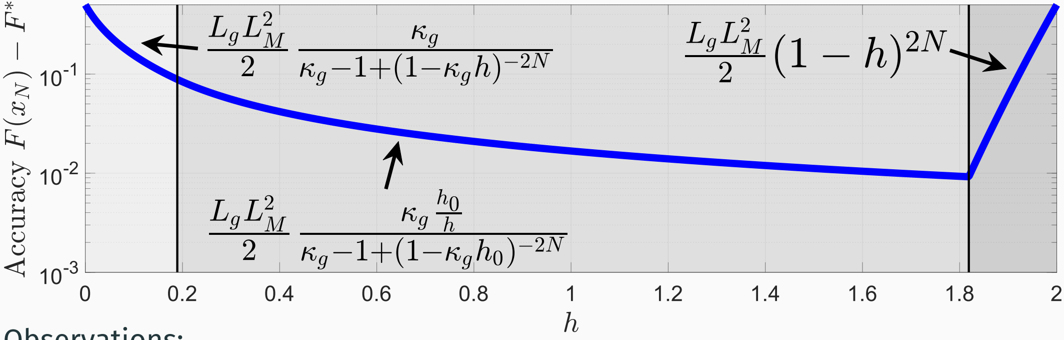
Observations:

Function class: $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx),\ g \text{ is } L_g\text{-smooth } \mu_g\text{-strongly convex},\ 0 \le ||M|| \le L_M\}$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L_g L_M^2}\nabla F(x_i)$

Parameters: $L_g = L_M = 1$, $\mu_g = 0.1$, $\kappa_g = \frac{\mu_g}{L_g}$, $||x_0 - x^*||^2 \le 1$, $h_0(\kappa_g, N)$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$



Observations:

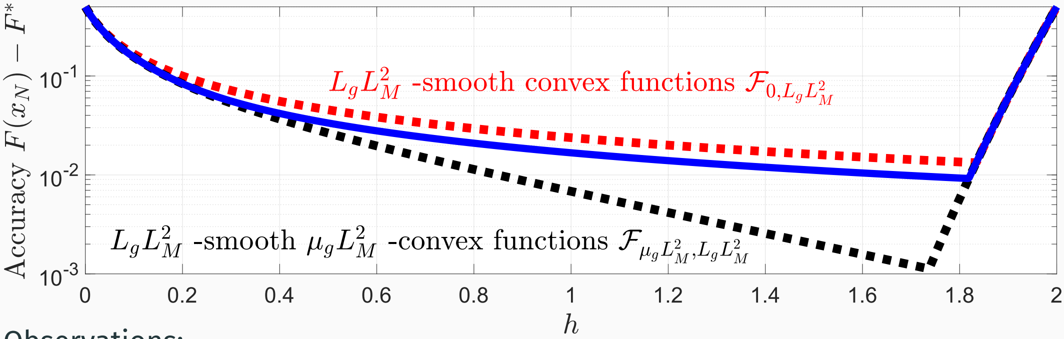
- 3 (identified) regimes;

Function class: $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx),\ g \text{ is } L_g\text{-smooth } \mu_g\text{-strongly convex},\ 0 \leq ||M|| \leq L_M\}$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L_g L_M^2}\nabla F(x_i)$

Parameters: $L_g = L_M = 1$, $\mu_g = 0.1$, $\kappa_g = \frac{\mu_g}{L_g}$, $||x_0 - x^*||^2 \leq 1$, $h_0(\kappa_g, N)$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$



Observations:

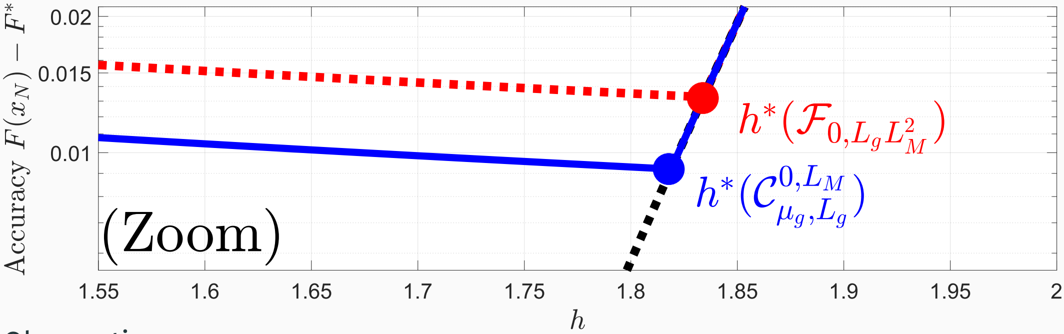
- 3 (identified) regimes;
- $\mathrm{Acc}(\mathcal{F}_{\mu_g L_M^2, L_g L_M^2}) \leq \mathrm{Acc}(\mathcal{C}^{0,L_M}_{\mu_g,L_g}) \leq \mathrm{Acc}(\mathcal{F}_{0,L_g L_M^2})$;

Function class: $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx), g$ is L_g-smooth μ_g-strongly convex, $0 \leq ||M|| \leq L_M\}$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L_g L_M^2} \nabla F(x_i)$

Parameters: $L_g = L_M = 1$, $\mu_g = 0.1$, $\kappa_g = \frac{\mu_g}{L_g}$, $||x_0 - x^*||^2 \leq 1$, $h_0(\kappa_g, N)$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$



Observations:

- 3 (identified) regimes;
- $\mathrm{Acc}(\mathcal{F}_{\mu_g L_M^2, L_g L_M^2}) \leq \mathrm{Acc}(\mathcal{C}^{0,L_M}_{\mu_g,L_g}) \leq \mathrm{Acc}(\mathcal{F}_{0,L_g L_M^2})$;
- Information on structure improves optimal step;

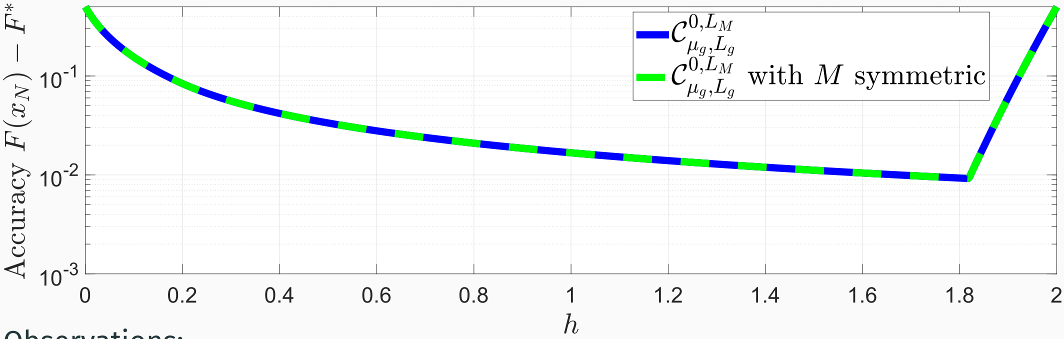
SIMPLEST PROBLEM $\min_x F(x)$ INVOLVING LINEAR MAPPING

Function class: $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx)$, g is L_g-smooth μ_g-strongly convex, $0 \leq ||M|| \leq L_M\}$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L_g L_M^2} \nabla F(x_i)$

Parameters: $L_g = L_M = 1$, $\mu_g = 0.1$, $\kappa_g = \frac{\mu_g}{L_g}$, $||x_0 - x^*||^2 \leq 1$, $h_0(\kappa_g, N)$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$



Observations:

- 3 (identified) regimes;
- $\mathrm{Acc}(\mathcal{F}_{\mu_g L_M^2, L_g L_M^2}) \leq \mathrm{Acc}(\mathcal{C}^{0,L_M}_{\mu_g,L_g}) \leq \mathrm{Acc}(\mathcal{F}_{0,L_g L_M^2})$;
- Information on structure improves optimal step;
- Symmetry has no impact.

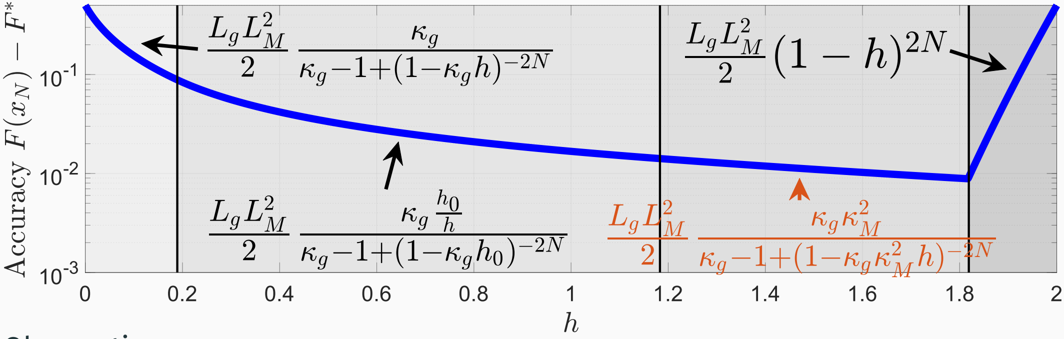
SIMPLEST PROBLEM $\min_x F(x)$ INVOLVING LINEAR MAPPING ($\mu_M > 0$)

Function class: $\mathcal{S}^{\mu_M, L_M}_{\mu_g, L_g} = \{F(x) = g(Mx),\ g$ is L_g-smooth μ_g-strongly convex, $\mu_M \leq ||M|| \leq L_M,\ M = M^T\}$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L_g L_M^2} \nabla F(x_i)$

Parameters: $L_g = L_M = 1$, $\mu_g = 0.1$, $\mu_M = 0.4$, $\kappa_g = \frac{\mu_g}{L_g}$, $\kappa_M = \frac{\mu_M}{L_M}$, $||x_0 - x^*||^2 \leq 1$, $h_0(\kappa_g, N)$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$



Observations:

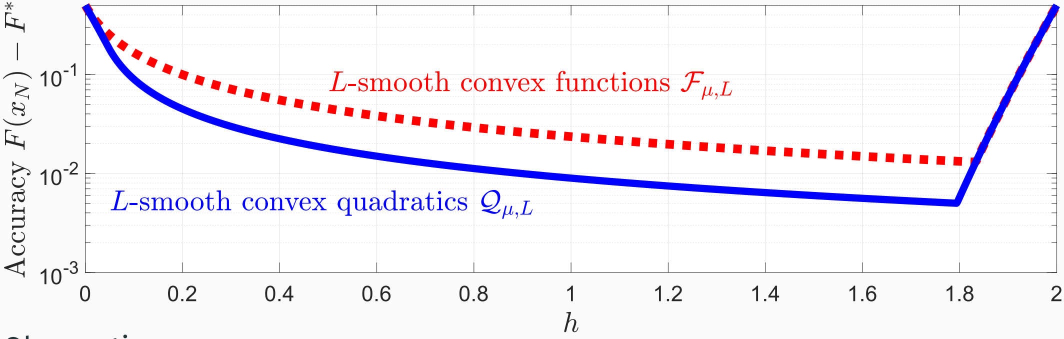
- 4 (identified) regimes;
- $\mathrm{Acc}(\mathcal{F}_{\mu_g L_M^2, L_g L_M^2}) \leq \mathrm{Acc}(\mathcal{S}^{\mu_M, L_M}_{\mu_g, L_g}) \leq \mathrm{Acc}(\mathcal{F}_{\mu_g \mu_M^2, L_g L_M^2})$;
- Information on structure improves optimal step.

Function class: $\mathcal{Q}_{\mu,L} = \{F(x) = \frac{1}{2}x^T Qx,\ \mu I \preceq Q \preceq LI,\ Q = Q^T\}(= \mathcal{S}_{1,1}^{\sqrt{\mu},\sqrt{L}})$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L}\nabla F(x_i)$

Notations/parameters: $L = 1,\ \mu = 0,\ ||x_0 - x^*||^2 \leq 1$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$



Observations:

- 3 (identified) regimes;
- $\mathrm{Acc}(\mathcal{Q}_{\mu,L}) \leq \mathrm{Acc}(\mathcal{F}_{\mu,L})$;
- Information on structure improves optimal step.

Analyzing a sophisticated algorithm

Problem: $\min_x f(x) + g(Mx)$ where f, g convex, proximable, $0 \leq ||M|| \leq L_M$

Chambolle-Pock (CP) method: $\begin{cases} x_{i+1} & = \mathrm{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} & = \mathrm{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$

Analyzing a sophisticated algorithm

Problem: $\min_x f(x) + g(Mx)$ where f, g convex, proximable, $0 \leq ||M|| \leq L_M$

Chambolle-Pock (CP) method: $\begin{cases} x_{i+1} & = \operatorname{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} & = \operatorname{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$

> Convergence results available but implicit and technical assumptions

Problem: $\min_x f(x) + g(Mx)$ where f, g convex, proximable, $0 \leq ||M|| \leq L_M$

Chambolle-Pock (CP) method: $\begin{cases} x_{i+1} & = \operatorname{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} & = \operatorname{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$

Convergence results available but implicit and technical assumptions

Examples of assumptions:

- Existence of sets B_1 and B_2 sufficiently large to contain all iterations x_i and u_i [Chambolle and Pock, 2011];
- Bound depending on the actual instance of the problem [Chambolle and Pock, 2016];
- Bound depending on the distance from initial to last point [Amir Beck, 2022].

Problem: $\min_x f(x) + g(Mx)$ where f, g convex, proximable, $0 \leq ||M|| \leq L_M$

Chambolle-Pock (CP) method: $\begin{cases} x_{i+1} &= \mathrm{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} &= \mathrm{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$

Convergence results available but implicit and technical assumptions

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- Bound depending on the distance from initial to last point [Amir Beck, 2022].

PEP uses classical explicit assumptions to unify performance guarantees.

Explicit assumption and performance criterion of choice

Problem: $\min_x f(x) + g(Mx)$ where f, g convex, proximable, $0 \leq ||M|| \leq L_M$, bounded subgradient

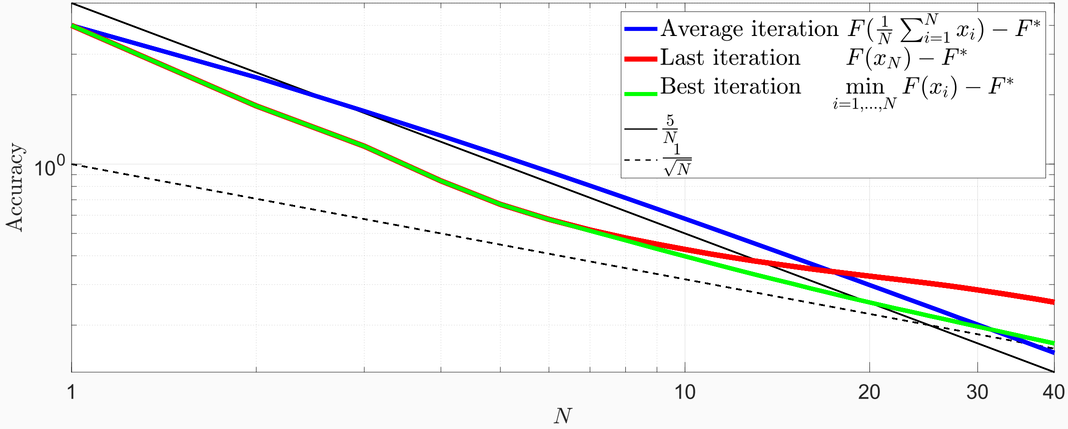
Chambolle-Pock (CP) method: $\begin{cases} x_{i+1} &= \text{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} &= \text{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$

Problem: $\min_x f(x) + g(Mx)$ where f, g convex, proximable, $0 \leq ||M|| \leq L_M$, bounded subgradient

Chambolle-Pock (CP) method:
$$\begin{cases} x_{i+1} &= \text{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} &= \text{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$$

Parameters: $L_M = \tau = \sigma = 1$, $\mu_M = 0$, $||x_0 - x^*|| \leq 1$, $||u_0 - u^*||^2 \leq 1$, $F(x) = f(x) + g(Mx)$

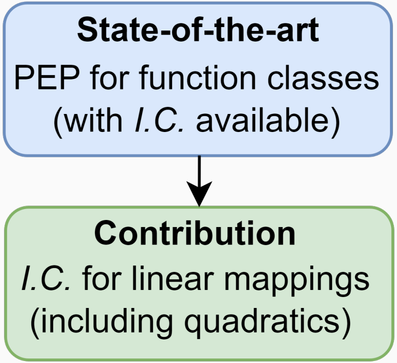
Worst-case performance of (CP) for varying number of iterations N



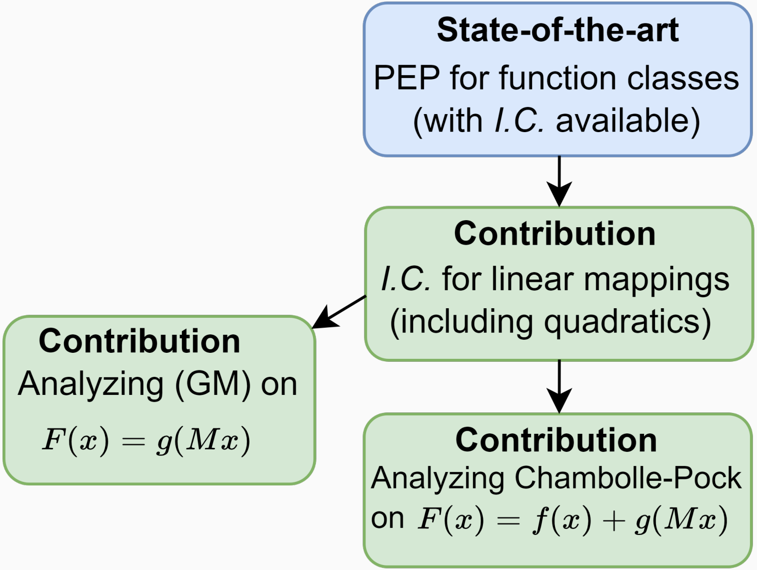
State-of-the-art

PEP for function classes
(with *I.C.* available)

I.C.: interpolation conditions

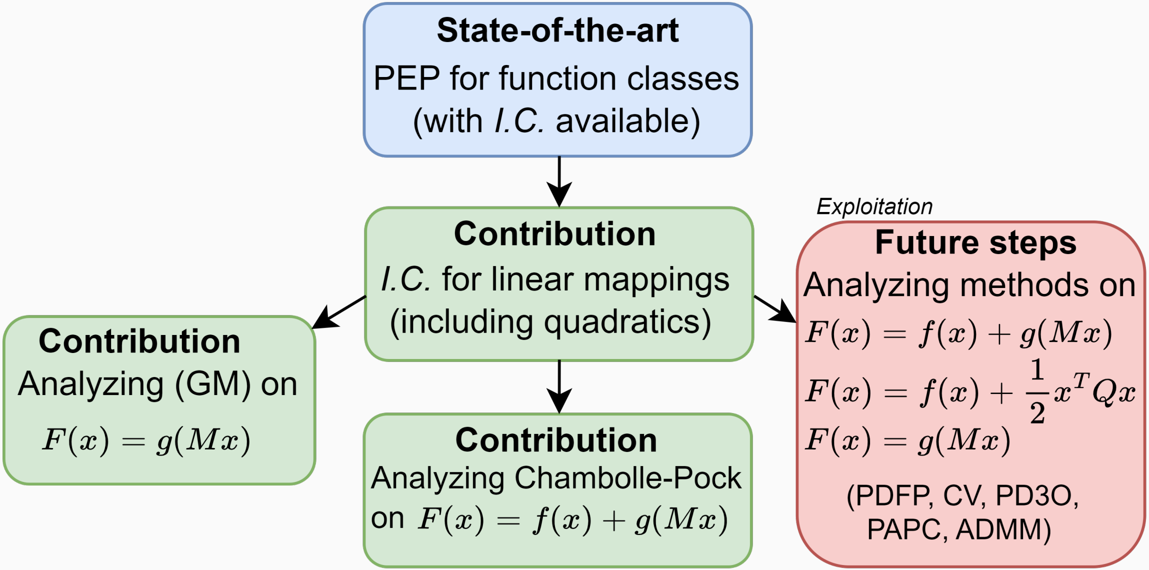


I.C.: interpolation conditions

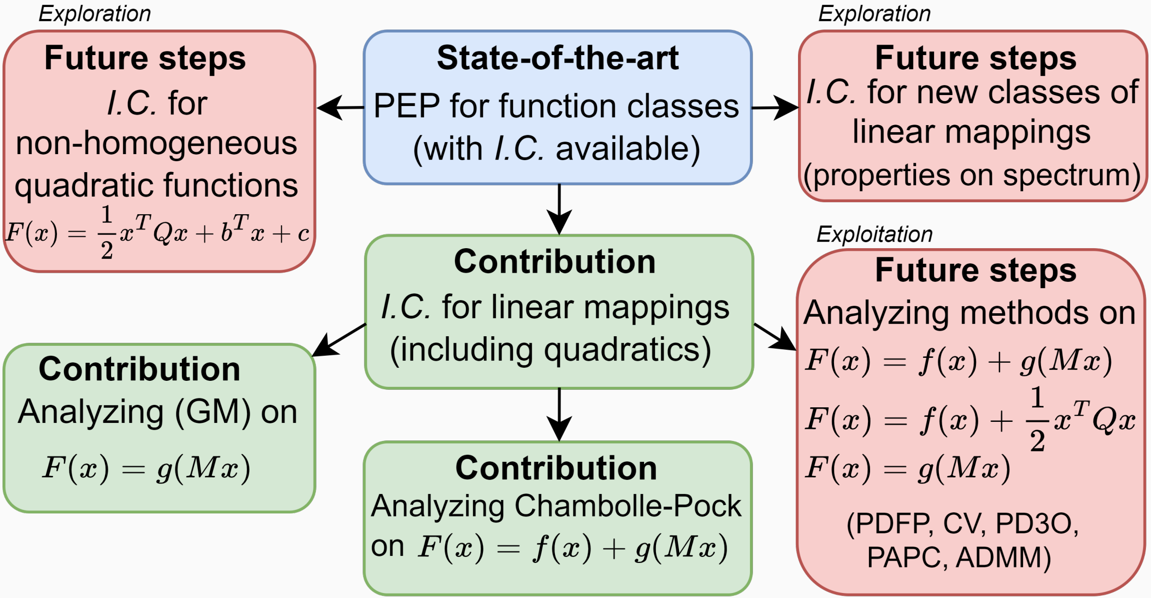


I.C.: interpolation conditions

PEP to analyze functions involving linear mapping



I.C.: interpolation conditions



I.C.: interpolation conditions

Worst-case performance of (GM) on $\mathcal{F}_{\mu,L}$ and $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$

Worst-case performance of (GM) on $\mathcal{F}_{\mu,L}$

$$F(x_N) - F^* \leq \frac{LR^2}{2} \max\left\{ \frac{\kappa}{\kappa - 1 + (1-\kappa h)^{-2N}}, (1-h)^{2N} \right\}$$

where $\kappa = \frac{\mu}{L}$.

Worst-case performance of (GM) on $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$

$$F(x_N) - F^* \leq \frac{L_g L_M^2 R^2}{2} \max\left\{ \frac{\kappa_g \alpha}{\kappa_g - 1 + (1-\kappa_g \alpha h)^{-2N}}, (1-h)^{2N} \right\}$$

where $\kappa_g = \frac{\mu_g}{L_g}$, $\kappa_M = \frac{\mu_M}{L_M}$ and $\alpha = \mathrm{proj}_{[\kappa_M^2, 1]}\left(\frac{h_0}{h}\right)$ for h_0 solution of

$$\begin{cases} (1-\mu_g)(1-\mu_g h_0)^{2N+1} = 1 - (2N+1)\mu_g h_0 \\ 0 \leq h_0 \leq \frac{1}{\mu_g}. \end{cases}$$

Worst-case functions of (GM) on $\mathcal{F}_{\mu,L}$ and $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$

Let $q(x) = \frac{1}{2}x^2$ and $\ell_{\mu,h}(x) = \begin{cases} \frac{\mu}{2}x^2 + (1-\mu)\tau_{\mu,h}|x| - \left(\frac{1-\mu}{2}\right)\tau_{\mu,h}^2 & \text{if } |x| \geq \tau_{\mu,h}, \\ \frac{1}{2}x^2 & \text{else,} \end{cases}$ where $\tau_{\mu,h} = \frac{\mu}{\mu-1+(1-\mu h)^{-2N}}$.

Worst-case functions of (GM) on $\mathcal{F}_{\mu,L}$ ($\kappa = \frac{\mu}{L}$) [Taylor, Hendrickx, Glineur, 2017]

$Lq(x) = \frac{L}{2}x^2$ and $L\ell_{\kappa,h}(x) = \begin{cases} \frac{\mu}{2}x^2 + (L-\mu)\tau_{\kappa,h}|x| - \left(\frac{L-\mu}{2}\right)\tau_{\kappa,h}^2 & \text{if } |x| \geq \tau_{\kappa,h}, \\ \frac{L}{2}x^2 & \text{else,} \end{cases}$

where $\tau_{\kappa,h} = \frac{\kappa}{\kappa-1+(1-\kappa h)^{-2N}}$.

Worst-case functions of (GM) on $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$ ($\kappa_g = \frac{\mu_g}{L_g}$, $\kappa_M = \frac{\mu_M}{L_M}$) [Bousselmi, Hendrickx, Glineur, 2023]

$L_g L_M q(x) = \frac{L_g}{2}(L_M x)^2$ and

$$L_g L_M M^2 \ell_{\kappa_g,M^2h}(x) = \begin{cases} \frac{\mu_g}{2}(L_M M x)^2 + (L_g-\mu_g)L_M M\tau_{\kappa_g,M^2h}|L_M M x| - \left(\frac{L_g-\mu_g}{2}\right)L_M M\tau_{\kappa_g,M^2h}^2 & \text{if } |x| \geq \tau_{\kappa_g,M^2h}, \\ \frac{L_g}{2}(L_M M x)^2 & \text{else,} \end{cases}$$

where $\tau_{\kappa_g,M^2h} = \frac{\kappa_g}{\kappa_g-1+\left(1-\kappa_g M^2 h\right)^{-2N}}$ and $M = \mathrm{Proj}_{[\kappa_M,1]}\left(\sqrt{\frac{h_0}{h}}\right)$.

Practical problems involving linear mappings

- ℓ_p-regularized robust regression ($p = 1, 2$)

$$\min_x ||Mx - b||_1 + ||x||_p^p,$$

- ℓ_1-constrained least squares

$$\min_x \; ||Mx - b||_2^2,$$
$$||x||_1 \leq c,$$

- Basis pursuit

$$\min_x \; ||x||_1,$$
$$Mx = c,$$

- Total variation deblurring

$$\min_x \; ||M_1x - b||_2^2 + ||M_2x||_1,$$

- Resource allocation

$$\min_x \; F(x),$$
$$Mx = c.$$

Methods with linear mappings (solve $\min_x f(x)+g(Mx)+h(x)$, g, h proximable and f smooth)

- **Primal-Dual Fixed Point (PDFP)** [Chen et al., 2016]

$$\begin{cases} \tilde{x} = \operatorname{prox}_{\tau h}\left(x_i - \tau \nabla f(x_i) - \tau M^T u_i\right), \\ u = \operatorname{prox}_{\sigma g^*}\left(u_i + \sigma M \tilde{x}\right), \\ x = \operatorname{prox}_{\tau h}\left(x_i - \tau \nabla f(x_i) - \tau M^T u\right), \\ x_{i+1} = x_i + \rho_i (x - x_i), \\ u_{i+1} = u_i + \rho_i (u - u_i). \end{cases}$$

- **Condat-Vu (CV)** [Condat, 2013, Vu, 2013]

$$\begin{cases} x = \operatorname{prox}_{\tau h}\left(x_i - \tau \nabla f(x_i) - \tau M^T u_i\right), \\ u = \operatorname{prox}_{\sigma g^*}\left(u_i + \sigma M(2x - x_i)\right), \\ x_{i+1} = x_i + \rho_i (x - x_i), \\ u_{i+1} = u_i + \rho_i (u - u_i). \end{cases}$$

- **Primal-Dual Three-Operator Splitting (PD3O)**

$$\begin{cases} x = \operatorname{prox}_{\tau h}\left(x_i\right), \\ u = \operatorname{prox}_{\sigma g^*}\left(u_i + \sigma M(2x - x_i - \tau \nabla f(x) - \tau M^T u_i)\right), \\ x_{i+1} = x_i + \rho_i (x - x_i - \tau \nabla f(x) - \tau M^T u), \\ u_{i+1} = u_i + \rho_i (u - u_i). \end{cases}$$

- **Alternating Direction Method of Multipliers (ADMM)** [Gabay and Mercier, 1976]:

$$\begin{cases} x_{i+1} \in \operatorname{arg\,min}_x f(x) + \frac{\rho}{2} M_1 x + M_2 y_i - c + \frac{1}{\rho} z_i^2, \\ y_{i+1} \in \operatorname{arg\,min}_y g(y) + \frac{\rho}{2} M_1 x_{i+1} + M_2 y - c + \frac{1}{\rho} z_i^2, \\ z_{i+1} = z_i + \rho (M_1 x_{i+1} + M_2 y_{i+1} - c), \end{cases}$$

solves $\min_x f(x) + g(y)$ s.t. $M_1 x + M_2 y = c$.

- **Proximal Alternating Predictor-Corrector (PAPC)** [Drori et al., 2015]

$$\begin{cases} p_{i+1} = x_i - \tau\left(M y_i + \nabla f(x_i)\right), \\ y_{i+1} = \operatorname{prox}_h\left(y_i + M^T p_{i+1}\right), \\ x_{i+1} = x_i - \tau\left(M y_{i+1} + \nabla f(x_i)\right), \end{cases}$$

solves $\min_x \max_y f(x) + x^T M y - h(y)$.

Proof of sufficiency in short

$$\text{Let } (X, Y, U, V) \text{ satisfying } \begin{cases} X^T V &= Y^T U, \\ Y^T Y &\preceq L^2 X^T X, \\ V^T V &\preceq L^2 U^T U, \end{cases} \text{ then,}$$

- **Step 1:** $\exists (X_R, Y_R, U_R, V_R)$ building the same Gram matrices, i.e.

$$(X\ V)^T (X\ V) = (X_R\ V_R)^T (X_R\ V_R),$$
$$(Y\ U)^T (Y\ U) = (Y_R\ U_R)^T (Y_R\ U_R),$$

and such that $\exists M_R$ with $\sigma_{\mathbf{max}}(M_R) \leq L$: $\begin{cases} Y_R = M_R X_R, \\ V_R = M_R^T U_R. \end{cases}$

- **Step 2:** If (X, Y, U, V) and (X_R, Y_R, U_R, V_R) build the same Gram matrices, then,

$$\exists R_1, R_2 \text{ unitary} : \begin{cases} (X_R\ V_R) = R_1 (X\ V), \\ (Y_R\ U_R) = R_2 (Y\ U), \end{cases} \quad \text{therefore,} \begin{cases} Y = \overbrace{R_2^T M_R R_1}^{M} X, \\ V = \overbrace{R_1^T M_R^T R_2}^{M^T} U. \end{cases}$$